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On a finite amplitude extension of geometric acoustics in a moving, inhomogeneous atmosphere

Stephen I. Warshaw

September 9, 1980

Lawrence Livermore Laboratory

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LAWRENCE LIVERMORE LABORATORY
University of California • Livermore, California • 94550

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ABSTRACT

The theory of linear geometric acoustics as developed by Blokhintsev is extended to the nonlinear acoustic regime, by applying the method of Riemann invariants to the propagation of acoustically small pulses along ray paths in a moving, inhomogeneous, thermoviscous medium. Blokhintsev's derivation is reviewed in detail, and is used to define an almost Galilean transformation of the nonlinear propagation equations from a quiescent to a moving medium. The nonlinear convective and dissipative processes are assumed to have a negligibly small effect on the ray paths. The differential equations of the ray paths and the propagation of the pulses are presented, and specialized to the example of a horizontally moving, gravitationally stratified atmosphere.

INTRODUCTION

The propagation of short acoustic pulses and their wavefronts over long distances in the atmosphere has provoked considerable interest during the past three decades or so, as evidenced by the copious theoretical literature on the subject. In addition to serving as a vehicle for expositions on different aspects of wave propagation theory, much of this published work treats atmospheric acoustic disturbances arising from natural and man-made sources: sonic booms from supersonic aircraft flight, nuclear and chemical explosions, jet noise, volcanic eruptions, storms, and other assorted violent phenomena. For the most part, these studies address problems of obtaining appropriate mathematical descriptions of the propagation process, of using such models to deduce the physical, chemical, and dynamical properties of the atmosphere, of predicting the effects of acoustic disturbances on the atmosphere (or vice versa), and so on. In this literature, a large variety of theoretical techniques are employed, and these may be loosely characterized as follows: (a) linear geometrical acoustics (rays and ray tubes), (b) linear harmonic wave theory (infrasound, acoustic-gravity waves, frequency-dependent modal analyses, Fourier methods), (c) nonlinear, finite amplitude wave theory, (d) weak-shock propagation theory, and (e) perturbation expansion methods. Representative reviews and bibliographies can be found in Refs.1–5 and 34; this list is by no means exhaustive.

From the standpoint of calculations of the long-range propagation of acoustic pulses in real atmospheres, the use of these techniques has so far not proven entirely satisfactory. The linear theories cannot account for the progressive alterations in the shape of the pulses as they propagate, while the significant application of the more complex nonlinear theories has generally been restricted to plane or spherical wavefront propagation in uniform, stationary atmospheres. During the past few years, a fresh theoretical approach has emerged, which is loosely referred to as nonlinear geometric acoustics, and which appears to have overcome these drawbacks. Its basic ideas are described succinctly by Ostrovsky.[6] In this paper, we present a simple variant of this method, which we apply to the problem of describing the propagation of acoustic pulses in moving, inhomogeneous atmospheres. Our idea is to use linear geometric acoustics to determine the propagation of the wavefront associated with the acoustic pulses, and then apply finite amplitude methods to determine in detail how the shape of the acoustic pulse changes as it is carried along the associated ray tubes. In this context, nonlinear effects appear as essential corrections to the linear theory, in what is hoped to be a proper manner.

This concept is not new, but our specific implementation of it appears to be. Previous work done along these lines has addressed the problem of calculating the effects of moving, stratified atmospheres on the nonlinear propagation of sonic boom pulses generated by aircraft in supersonic flight. Friedman, Kane and Sigalla[7] and Guiraud[8] have developed comprehensive treatments, in which the equations governing the propagation of the pulses along rays are produced. In these cases the rays are determined to be trajectories along which the sonic boom shock front propagates, in various approximate senses. Guiraud also outlines the historical development of this approach, which owes much to Whitham.[2] Hayes and Runyan[9] assume propagation along ray tubes in the manner of linear geometric acoustics, and then apply nonlinear corrections. Of these treatments, only Guiraud has indicated (albeit cursorily) how dissipative effects are to be included. Varley and Cumberbatch[10] discuss large amplitude shockless acoustic pulses travelling in stratified media. Their work is of interest for the useful insights it provides in relation to our work.

The main advantage of the use of nonlinear methods along the ray tubes is its relative simplicity, which arises from assuming that the calculation of the location of the wavefront (i.e., the ray tubes) and the calculation of the nonlinear propagation can be partially decoupled. (As Friedman et al.[7] correctly point out, this is not completely correct, particularly in situations where acoustic rays can cross. Thus, the theory presented here is not strictly applicable in such regions.) Also, if the ambient atmosphere varies smoothly enough that the ray tubes do not have sharp bends, the nonlinear propagation along the tubes can be calculated almost one-dimensionally. This requires that the acoustic ray radius of curvature be substantially larger than the mean spatial thickness of the pulse, a condition that is almost always satisfied by audible sound pulses in the atmosphere.

The disadvantage of this approach is that communication is not allowed between adjacent ray tubes, and this introduces a distortion into the propagation process. To minimize this drawback, at least two conditions on the pulse and its wavefront should be satisfied. First, the propagating wavefront should be stable in the sense that it does not change shape appreciably as the wave propagates. This corresponds to the situation in optics in which a wave has attained the Fraunhofer zone: all wave and wavefront irregularities have then been diffracted out. Second, transverse diffusion effects in the pulse should be negligible, or almost so, since molecular transport processes do not respect ray tube boundaries. Alternatively, one could, instead, require that the net momentum transfer due to diffusion across the ray tube walls balance out to negligible proportions, which can be guaranteed to some extent by requiring that the tangential (i.e., transverse) gradients in the overall pulse be small. It is evident from all these considerations that we are here describing a pulsed wave that is locally planar, as, e.g., a spherical wavefront of large radius.

In this paper we shall first review some details of linear geometric acoustics that are relevant to our approach, then show how dissipative and nonlinear effects are to be incorporated, and how the approximations required to do so are made.

LINEAR ACOUSTICS

In this and the following section, we shall derive the differential equations that determine the acoustic rays and the propagation of acoustic disturbances along these rays, in the linear approximation. We shall closely reproduce the essential details of the treatment of Blokhintsev,[11,12] which is apparently the earliest work that rigorously develops the linear acoustics of steady, but otherwise arbitrarily varying, media. The actual ray equations will be derived in a later section.

We begin with the standard equations[13] for a non-reacting, compressible, viscous, heat-conducting fluid, which determine the flow of mass, momentum, and entropy, respectively:

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0 \quad , \qquad (1)$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{1}{\rho}\nabla p = \mathbf{g} + \frac{1}{\rho}\nabla\overleftrightarrow{\sigma} \quad , \qquad (2)$$

and

$$\frac{\partial s}{\partial t} + \mathbf{v} \cdot \nabla s = \frac{1}{\rho T} [(\overleftrightarrow{\sigma}\nabla) \cdot \mathbf{v} + \nabla \cdot (\kappa \nabla T)], \quad (3)$$

where $\mathbf{v}$ is the fluid velocity, $\mathbf{g}$ is the external body force per unit mass (e.g., gravity), ρ, p, s, and T are the density, pressure, specific entropy, and temperature of the fluid, and $\overleftrightarrow{\sigma}$ is the viscosity stress tensor[13] defined (in Cartesian components) by

$$\sigma_{ij} = \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right) + \zeta \delta_{ij}\frac{\partial v_k}{\partial x_k} . \quad (4)$$

Here, κ, η, and ζ are the coefficients of heat conduction, shear and bulk viscosity, and the repeated-index summation convention is understood, viz., $\partial v_k/\partial x_k = \nabla \cdot \mathbf{v}$.

These equations are augmented by an as yet unspecified equation of state that relates the thermodynamic variables for any given fluid. The propriety of employing equilibrium thermodynamic concepts in the specification of a *dynamic* situation is discussed by Landau and Lifshitz.[14] Implicit in this employment is the assumption that the fluid is locally in reversible thermodynamic equilibrium; it is this approximation that allows the entropy of the fluid to be specified as a definite state function, so that Eq. (3) also describes the flow of energy in the fluid. Thus, to the extent this assumption is valid, we may replace differentials of s with those of other thermodynamic variables, by making use of relations such as

$$T\delta s = \delta\epsilon - \frac{p}{\rho^2}\delta\rho , \quad (5)$$

where ϵ is the specific internal energy of the fluid, and δ is any relevant differential operator.

The key defining feature of a local acoustic disturbance is that its magnitude is so small that it contributes negligibly to the motion of the ambient fluid, which it perturbs. Accordingly, we suppose now that the fluid properties f can be regarded as the sum of an ambient f_0 plus a local acoustic perturbation f_1, where f is any of the fluid variables. Then Eqs. (1)–(4) are assumed to hold when $f = f_0$, and when $f = f_0 + f_1$. Another way to regard this situation is to consider the ambient fluid to be stable, in the sense that its motion is independent of whether a perturbation is present or not. (This assumption is much more general than the customary one in which the ambient fluid motion and properties are assumed steady.) The equations of motion for the ambient fluid are thus

$$\frac{\partial \rho_0}{\partial t} + \mathbf{v}_0 \cdot \nabla\rho_0 + \rho_0 \nabla \cdot \mathbf{v}_0 = 0 , \quad (6)$$

$$\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \frac{1}{\rho_0}\nabla p_0 = \mathbf{g} + \frac{1}{\rho_0}\nabla\overleftrightarrow{\sigma}_0 , \quad (7)$$

and

$$\frac{\partial s_0}{\partial t} + \mathbf{v}_0 \cdot \nabla s_0 = \frac{1}{\rho_0 T_0}[(\overleftrightarrow{\sigma}_0\nabla)\cdot \mathbf{v}_0 + \kappa\nabla^2 T_0] , \quad (8)$$

where $\overleftrightarrow{\sigma}_0$ is $\overleftrightarrow{\sigma}$ when $\mathbf{v} = \mathbf{v}_0$, and κ is assumed to be essentially independent of position. For steady (or slowly varying) ambient fluids, the partial time derivatives may be dropped, and when $\mathbf{v}_0 = 0$ (quiescent ambient fluid), Eq. (7) becomes the familiar condition of hydrostatic equilibrium, while Eq. (8) reduces to one form of the diffusion equation of heat flow.

At this point it is worth noting that the acoustic assumption tacitly requires the ambient fluid to be mechanically stable. (That is, the disturbance should not trigger a major motion from a state of unstable equilibrium.) This is certainly true of the real atmosphere.[3] The motions associated with restoring the atmosphere to equilibrium after the passage of a disturbance are Väisälä-Brunt oscillations whose periods are much longer than those of the acoustic perturbations we intend to consider, and whose wavelengths are comparable to atmospheric scale heights (cf., e.g., Beer[3]). By analogy from simple resonant systems we would not expect these oscillations to be generated to any significant magnitude by short acoustic pulses. It is interesting to note that the problem of determining the amplitude of these oscillations from such acoustic sources does not yet seem to have been adequately treated in the literature.

Returning to our equations, the motion of the fluid with the imposed acoustic perturbation is given by Eqs. (1)–(4), with $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$, and so on. After performing this substitution, we make use of Eqs. (6)–(8) to remove the terms involving the motion of the ambient fluid. The resultant set of equations for the perturbation is much too complicated to work with. To make this system more tractable, we keep the remaining dominant terms, i.e., those which are first order in the perturbation variable, and discard all terms containing the dissipative parameters. This implies that the perturbation is propagated with negligible damping. The result is then Blokhintsev's system of linear equations of motion of the acoustic perturbation:

$$\frac{\partial \rho_1}{\partial t} + \mathbf{v}_1 \cdot \nabla\rho_0 + \mathbf{v}_0 \cdot \nabla\rho_1 + \rho_0\nabla \cdot \mathbf{v}_1 + \rho_1\nabla \cdot \mathbf{v}_0 = 0 \ , \qquad (9)$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + \mathbf{v}_0 \cdot \nabla\mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla\mathbf{v}_0 + \frac{1}{\rho_0}\nabla p_1 - \frac{\rho_1}{\rho_0^2}\nabla p_0 = 0 \ , \qquad (10)$$

and

$$\frac{\partial s_1}{\partial t} + \mathbf{v}_0 \cdot \nabla s_1 + \mathbf{v}_1 \cdot \nabla s_0 = 0 \ . \qquad (11)$$

By making use of a vector identity, we can put Eq. (10) in a form which does not depend on the coordinate system to be Cartesian:

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\nabla \times \mathbf{v}_0) \times \mathbf{v}_1 + (\nabla \times \mathbf{v}_1) \times \mathbf{v}_0 + \nabla(\mathbf{v}_0 \cdot \mathbf{v}_1) + \frac{1}{\rho_0}\nabla p_1 - \frac{\rho_1}{\rho_0^2}\nabla p_0 = 0 . \qquad (12)$$

These equations apply to a fluid in which there exist ambient vortices and entropy gradients; there is no restriction that the perturbation itself be irrotational, or that the ambient fluid be free of dissipative effects, thermal sources, or sinks.

The above equations must be augmented by a thermodynamic equation of state, which provides an additional relation between the acoustic variables needed to make them complete. Let us assume that the pressure is a function of the density and entropy, $p = p(\rho,s)$. For small perturbations about the ambient values this may be expanded as

$$p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_s(\rho - \rho_0) + \left(\frac{\partial p}{\partial s}\right)_\rho(s - s_0) \qquad (13)$$

or

$$\Delta p = \left(\frac{\partial p}{\partial \rho}\right)_s\Delta\rho + \left(\frac{\partial p}{\partial s}\right)_\rho \Delta s \ . \qquad (14)$$

Note that this relationship is sufficiently general that it applies also to differentials of the variables. Thus, e.g., we can have

$$\nabla p = \left(\frac{\partial p}{\partial \rho}\right)_s\nabla\rho + \left(\frac{\partial p}{\partial s}\right)_\rho\nabla s \ . \qquad (15)$$

Hence, if the acoustic variables are regarded as incremental changes on the ambient values, we can identify p_1 with Δp, ρ_1 with $\Delta\rho$, s_1 with Δs, and obtain an "equation of state" for the acoustic perturbation in the form

$$p_1 = h_0 s_1 + a_0^2\rho_1 \ , \qquad (16)$$

where $h_0 = (\partial p/\partial s)_\rho$ and $a_0^2 = (\partial p/\partial \rho)_s$; a_0 is the ambient adiabatic speed of sound. Equation (16) is valid provided, as we have so far assumed, that dissipative effects for the perturbation are negligible. In a later section, this equation will be modified.

LINEAR GEOMETRIC ACOUSTICS

We now use Blokhintsev's equations to calculate how a wavefront carrying an acoustic pulse propagates into a steadily moving ambient medium. Since these equations are linear in the acoustic variables, it is convenient to work with sinusoidal waves, for which the wavefronts are surfaces of constant phase.

At first glance this approach appears to have little to do with our pulses, but in fact it has much to do with them. On one hand, the linearity of the equations allows the pulses to be represented by Fourier superpositions of high-frequency sinusoidal waves, and so we would need only deal with a representative monochromatic component. On the other hand, the pulses are more realistically represented by a moving surface of discontinuity in the acoustic quantities, on the forward side of which the ambient fluid is undisturbed. Now, this latter method of representation was developed very fully by Kline[15] and Kline and Kay[16] for electromagnetic disturbances, and consequently employed by Heller[17] in the hydrodynamic case. Heller's equations for the surface of discontinuity proved identical to the equation for the wave surface obtained in the sinusoidal case. Kline,[15] Kline and Kay,[16] and Whitham[18] indicate that, in the limit of high frequencies, either approach produces equivalent results; Whitham shows that the equations for the surfaces and the associated amplitudes obtained by each method are formally identical.

We therefore assume that the acoustic variables in Blokhintsev's equations each take the form $Ae^{i(\omega t - k\theta)}$, where the amplitude A depends on both the coordinates and the time, ω and k are the reference constant frequency and wave number, and the desired surfaces of constant phase are, at any given time, described by θ = constant. Since the ambient fluid is assumed steady, θ is evidently a function only of the coordinates. The temporal progression of any given surface of constant phase is described by $\omega t - k\theta$ = constant. We thus replace p_1 by $p_1 e^{i(\omega t - k\theta)}$, and so on, carry out the indicated differentiation, and factor out the common exponential. This procedure is formally equivalent to replacing the operators ∇ and $\partial/\partial t$ in Eqs. (9), (11), and (12) by $(\nabla - ik\nabla\theta)$ and $(\partial/\partial t + i\omega)$, respectively, and interpreting p_1, ρ_1, etc., as sinusoidal amplitudes. The result, after some rearranging of terms, is

$$\frac{\partial \rho_1}{\partial t} + \mathbf{v}_0 \cdot \nabla\rho_1 + \mathbf{v}_1 \cdot \nabla\rho_0 + \rho_0 \nabla \cdot \mathbf{v}_1 + \rho_1 \nabla \cdot \mathbf{v}_0 = -ik(q\rho_1 - \rho_0 \mathbf{v}_1 \cdot \nabla\theta) \ , \quad (17)$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + (\nabla \times \mathbf{v}_0) \times \mathbf{v}_1 + (\nabla \times \mathbf{v}_1) \times \mathbf{v}_0 + \nabla(\mathbf{v}_0 \cdot \mathbf{v}_1) + \frac{1}{\rho_0}\nabla p_1 - \frac{\rho_1}{\rho_0^2}\nabla p_0 = -ik\left(q\mathbf{v}_1 - \frac{p_1}{\rho_0}\nabla\theta\right) \ , \quad (18)$$

and

$$\frac{\partial s_1}{\partial t} + \mathbf{v}_0 \cdot \nabla s_1 + \mathbf{v}_1 \cdot \nabla s_0 = -ikqs_1 \ . \quad (19)$$

In the process, we have defined a frequently occurring quantity by

$$q = q_0 - \mathbf{v}_0 \cdot \nabla\theta \quad (20)$$

where $q_0 = \omega/k$ is a reference constant velocity, and in obtaining Eq. (18) from Eq. (12) we used the identity for the vector triple cross product. Next, we further condense the notation by representing the expressions on the left side in Eqs. (17), (18), and (19) by $-F$, $-G$, and $-H$, respectively. Then

$$q\rho_1 - \rho_0 \mathbf{v}_1 \cdot \nabla\theta = \frac{q_0 F}{i\omega} \ , \quad (21)$$

$$q\mathbf{v}_1 - \frac{p_1}{\rho_0}\nabla\theta = \frac{q_0 G}{i\omega} \ , \quad (22)$$

and

$$qs_1 = \frac{q_0 H}{i\omega} \quad (23)$$

are the corresponding results. And, from Eq. (16), we know that

$$\rho_1 = \frac{1}{a_0^2}(p_1 - h_0 s_1) \ . \quad (24)$$

As indicated by Whitham,[17] we can now express the acoustic amplitudes as an asymptotic series in powers of $1/i\omega$. Thus, we write each of the acoustic variables in the form $f_1 = \Sigma_m g_m(i\omega)^{-m}$, where f_1 is p_1, $\mathbf{v}_1$, and s_1, and the coefficients g_m of various orders are correspondingly π_m, $\boldsymbol{\phi}_m$, and σ_m. Since F, G, and H are each linear in the amplitudes, we also have the expansions where the f are these, also, and the corresponding g_m are F_m, G_m, and H_m. These expressions are now inserted into Eqs. (21)–(23), and coefficients of like powers of $i\omega$ equated. The result is the following set of recursive relations between successive orders of approximation:

$$\frac{q}{a_0^2}(\pi_m - h_0\sigma_m) - \rho_0\boldsymbol{\phi}_m \cdot \nabla\theta = q_0F_{m-1} \ , \quad (25)$$

$$q\boldsymbol{\phi}_m - \frac{\pi_m}{\rho_0}\nabla\theta = q_0\mathbf{G}_{m-1} \ , \quad (26)$$

and

$$q\sigma_m = q_0H_{m-1} \ . \quad (27)$$

For m = 0 the right-hand quantities are zero, and we obtain, for the dominant first approximation, the equations

$$\frac{q\pi_0}{a_0^2} - \rho_0\boldsymbol{\phi}_0 \cdot \nabla\theta = 0 \quad (28)$$

and

$$q\boldsymbol{\phi}_0 - \frac{\pi_0}{\rho_0}\nabla\theta = 0 \ . \quad (29)$$

Note that this is the same result that would have been obtained from Eqs. (21)–(24) in the limit of infinite frequency. It is evident that the propagation in this approximation is isentropic: $\sigma_0 = 0$, and $\rho_1 = \pi_0/a_0^2$. It is also longitudinal in the sense that from Eq. (29), the acoustic perturbation fluid velocity is in the same direction as the wavefront normal, whose direction is given by $\nabla\theta$. Eliminating $\nabla\theta$ from these equations gives the familiar small-amplitude relation of linear acoustics:

$$\pi_0 = \rho_0 a_0 \phi_0. \quad (30)$$

On the other hand, the elimination of $\boldsymbol{\phi}_0$ yields what is generally referred to as the eikonal equation of linear geometric acoustics:

$$(\nabla\theta)^2 = \left(\frac{q}{a_0}\right)^2 \ , \quad (31)$$

or, from Eq. (20),

$$(\nabla\theta)^2 = \frac{1}{a_0^2}(q_0 - \mathbf{v}_0 \cdot \nabla\theta)^2 \ . \quad (32)$$

This is the differential equation for the surfaces of constant phase, which will be used later to obtain the equations for the acoustic rays.

An analogous equation was obtained by Heller[17] on the basis of moving surfaces of discontinuity; in his result, the velocity $\mathbf{v}_0$ in Eq. (32) is the total material velocity including the perturbation, i.e., $\mathbf{v}_0 + \mathbf{v}_1$. This is a physically more accurate procedure; Heller, however, did not obtain the discontinuity amplitudes.

Having obtained expressions for $\nabla\theta$, we now seek to obtain the leading acoustic amplitudes, which are embedded in F_0, $\mathbf{G}_0$, and H_0. On combining Eqs. (25) and (27), we get

$$\frac{q\pi_m}{a_0^2} - \rho_0\boldsymbol{\phi}_m \cdot \nabla\theta = q_0\left(F_{m-1} + \frac{h_0}{a_0^2}H_{m-1}\right). \quad (33)$$

It turns out that, because of Eq. (31), the linear system of Eqs. (26) and (33) has a vanishing determinant; this, indeed, is the same determinant for Eqs. (28) and (29). To handle this situation, we take the scalar product of Eq. (26) with $\nabla\theta$, and make use of Eq. (31), to obtain

$$q\boldsymbol{\phi}_m \cdot \nabla\theta - \frac{\pi_m q^2}{\rho_0 a_0^2} = q_0\mathbf{G}_{m-1} \cdot \nabla\theta \ . \quad (34)$$

If we multiply Eq. (33) by q and Eq. (34) by ρ_0, we obtain, as one would expect, a pair of equations whose left-hand sides are identical, apart from a negative sign. From this we obtain the following combination of the factors of the right-hand sides:

$$qF_m + \frac{qh_0}{a_0^2}H_m + \rho_0\mathbf{G}_m \cdot \nabla\theta = 0 \ . \quad (35)$$

For m = 0, this is Blokhintsev's result.

By virtue of the way F_0, $\mathbf{G}_0$, and H_0 were defined, Eq. (35) is a linear, first order differential equation for the propagation of the acoustic amplitudes. This is not immediately apparent, however. To reduce Eq. (35) to an expression in just one of the acoustic variables, a considerable amount of vector and scalar algebra is involved, and we shall indicate only certain steps. We make use of all the leading approximation results (Eqs. (28)–(32)), as well as Eqs. (15) and (20), and the isentropic condition. At the same time, we use the fact that we are dealing with a steady fluid, so that no partial time derivatives of the ambient variables are needed. From Eq. (29) which relates $\boldsymbol{\phi}_0$ and $\nabla\theta$, Eq. (35) is put in the form (m = 0)

$$F_0 + \frac{h_0}{a_0^2}H_0 + \frac{\rho_0^2}{\pi_0}G_0 \cdot \boldsymbol{\phi}_0 = 0 \ . \tag{36}$$

By their definitions, and using the fact that $\sigma_0 = 0$, we have

$$-F_0 = \frac{\partial}{\partial t}\left(\frac{\pi_0}{a_0^2}\right) + \mathbf{v}_0 \cdot \nabla\left(\frac{\pi_0}{a_0^2}\right) + \boldsymbol{\phi}_0 \cdot \nabla\rho_0 + \rho_0\nabla \cdot \boldsymbol{\phi}_0 + \frac{\pi_0}{a_0^2}\nabla \cdot \mathbf{v}_0 \ , \tag{37}$$

$$-H_0 = \boldsymbol{\phi}_0 \cdot \nabla s_0 \ , \tag{38}$$

and also

$$-G_0 \cdot \boldsymbol{\phi}_0 = \boldsymbol{\phi}_0 \cdot \frac{\partial \boldsymbol{\phi}_0}{\partial t} + \frac{1}{\rho_0}\boldsymbol{\phi}_0 \cdot \nabla\pi_0 - \frac{\pi_0}{\rho_0^2 a_0^2}\boldsymbol{\phi}_0 \cdot \nabla p_0 + Q \ , \tag{39}$$

where

$$Q = \boldsymbol{\phi}_0 \cdot (\nabla \times \mathbf{v}_0) \times \boldsymbol{\phi}_0 + \boldsymbol{\phi}_0 \cdot (\nabla \times \boldsymbol{\phi}_0) \times \mathbf{v}_0 + \boldsymbol{\phi}_0 \cdot \nabla(\mathbf{v}_0 \cdot \boldsymbol{\phi}_0) \ . \tag{40}$$

These items are inserted into Eq. (36). We then proceed to eliminate $\boldsymbol{\phi}_0$ by using Eq. (29) and by noting that, from Eqs. (29) and (30), $\boldsymbol{\phi}_0 = (\pi_0/\rho_0 a_0)\hat{\mathbf{n}}$ where $\hat{\mathbf{n}} = \nabla\theta/|\nabla\theta|$ is the unit normal to the wavefront surface in the direction of travel. We eliminate the combination $\mathbf{v}_0 \cdot \nabla\theta$ by making liberal use of Eq. (20).

The result, after collecting terms, is the following remarkable equation in π_0:

$$\frac{\partial\pi_0}{\partial t} + \mathbf{v}_s \cdot \nabla\pi_0 + \frac{1}{2}[\nabla \cdot \mathbf{v}_s - \mathbf{v}_s \cdot \nabla \ln(\rho_0 q a_0^2)]\pi_0 = 0 \ , \tag{41}$$

where we have defined

$$\mathbf{v}_s = \mathbf{v}_0 + a_0\hat{\mathbf{n}} \ . \tag{42}$$

Except for $\hat{\mathbf{n}}$, the quantity in the brackets is solely a function of the properties of the ambient fluid. This is essentially Blokhintsev's main result, apart from what are apparently typographical misplacements of factors of 2 in both of his publications.[11,12] It is evident that $\mathbf{v}_s$ is the velocity of propagation of the acoustic disturbance, represented by the amplitude π_0.

It now remains to demonstrate that $\mathbf{v}_s$ is also the velocity (speed and direction) with which the wavefront surface actually moves. Recall that the surfaces of constant phase are given by the function $\Phi = \omega t - k\theta$ = constant. If the assertion is true, then this function should satisfy the propagation condition

$$\frac{\partial\Phi}{\partial t} + \mathbf{v}_s \cdot \nabla\Phi = 0 \ . \tag{43}$$

On substituting for Φ, we obtain $\omega - k\mathbf{v}_s \cdot \nabla\theta = 0$, or $\mathbf{v}_s \cdot \nabla\theta = q_0$. With the use of Eqs. (20), (31), and (42), a little algebra shows this last relation is identically true, and so, therefore, is the assertion.

It is not usually appreciated that Blokhintsev's result, in common with most linear, small-amplitude acoustic theories, is also true for short disturbances of arbitrary profile, even though the ambient medium is moving and inhomogeneous. To see this, let us represent one such disturbance as a Fourier synthesis over a suitably high frequency domain, namely

$$p_1 = \int \pi_0 e^{i\Phi} d\omega \ , \tag{44}$$

where Φ is as given above, and we now identify π_0 as the spectral amplitude. On taking space and time derivatives in Eq. (44), we find that

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_s \cdot \nabla p_1 = \int \left[\frac{\partial\pi_0}{\partial t} + \mathbf{v}_s \cdot \nabla\pi_0 + i\pi_0\left(\frac{\partial\Phi}{\partial t} + \mathbf{v}_s \cdot \nabla\Phi\right)\right] e^{i\Phi} d\omega \ . \tag{45}$$

From Eq. (43) the imaginary terms in Eq. (45) vanish. This means that if Eq. (41) is multiplied by

$e^{i\Phi}$, and then integrated over ω, the following generalization of Blokhintsev's equation obtains:

$$\frac{\partial p_1}{\partial t} + \mathbf{v}_s \cdot \nabla p_1 + \frac{1}{2}\left[\nabla \cdot \mathbf{v}_s - \mathbf{v}_s \cdot \nabla \ln(\rho_0 q a_0^2)\right] p_1 = 0 , \quad (46)$$

where p_1 is the actual disturbance function, not the amplitude. In a similar manner, all the previous developments hold for disturbances of arbitrary profile. In particular, the small amplitude relations given by Eqs. (28)–(30) now take the forms

$$p_1 = \rho_0 a_0 v_1 = a_0^2 \rho_1 \quad (47)$$

and

$$\mathbf{v}_1 = \frac{p_1}{\rho_0 q} \nabla \theta = \frac{p_1}{\rho_0 a_0} \hat{\mathbf{n}} , \quad (48)$$

and these give the other acoustic variables.

The total kinetic energy of the ambient fluid carrying this disturbance is

$$T = \frac{1}{2}(\rho_0 + \rho_1)(\mathbf{v}_0 + \mathbf{v}_1)^2$$
$$= T_0 + (\rho_0 + \rho_1)\mathbf{v}_0 \cdot \mathbf{v}_1 + \frac{1}{2}(\rho_0 v_1^2 + \rho_1 v_0^2) , \quad (49)$$

where terms of order higher than the second in the acoustic variables have been dropped, and where T_0 is the ambient kinetic energy in the absence of the perturbation. We identify the kinetic energy of the acoustic disturbance with the time-averaged value of $T - T_0$; for each spectral component, this is

$$\langle T - T_0 \rangle = \frac{1}{2}\rho_0 \phi_0^2 + \frac{\pi_0}{a_0^2} \phi_0 \cdot \mathbf{v}_0 , \quad (50)$$

which, on making use of Eqs. (20), (29), and (30), becomes

$$\langle T - T_0 \rangle = \frac{\pi_0^2}{\rho_0 a_0^2}\left[\frac{q_0}{q} - \frac{1}{2}\right] . \quad (51)$$

For a motionless ambient fluid $q = q_0$, and we regain the familiar result for linear acoustics.

The mean internal energy associated with the perturbation, however, is a considerably more complex and delicate matter; this is discussed by Morfey,[19] and in other references quoted by him, in the context of general acoustic energy flows. Blokhintsev,[12] from considerations of thermodynamic perturbation expansions, obtains

$$\langle \epsilon_1 \rangle = \frac{\pi_0^2}{2\rho_0 a_0^2} , \quad (52)$$

which is also a well-known classical result.[13,20] Equations (51) and (52) then yield

$$\langle E \rangle = \frac{\pi_0^2 q_0}{\rho_0 a_0^2 q} . \quad (53)$$

On summing this result over the frequency range, we obtain the total acoustic energy carried by the disturbance as

$$E = \int \langle E \rangle d\omega = \frac{p_1^2 q_0}{\rho_0 a_0^2 q} . \quad (54)$$

From Eq. (20), we find that

$$\frac{q_0}{q} = 1 + \frac{v_n}{a_0} , \quad (55)$$

where v_n is the projection of $\mathbf{v}_0$ along the wavefront normal $\hat{\mathbf{n}}$.

It is this last term that distinguishes Blokhintsev's results from those obtained in the standard case of a motionless ambient fluid. Quite simply, it is due to the fact that the propagating wavefront moves faster or slower according as the flow is with or against the direction of its motion perpendicular to itself. In anticipation of the subsequent discussion on ray tubes, consider a segment of area $A_0\hat{\mathbf{n}}$ embedded in the wavefront or phase surface. During the propagation, the segment will sweep out a volume $(A_0\hat{\mathbf{n}}) \cdot \mathbf{v}_s = A_0(a_0 + v_n)$ per unit time. If the medium were motionless, this volume would be $A_0 a_0$. Blokhintsev's factor in Eq. (55) is just the ratio of these two volumes. If we regard these swept-out volumes as segments of ray tubes, then the ray tubes will be seen to expand or contract along their length relative to the motionless ambient case, much like a pleatless concertina. Some authors (e.g., Lighthill[5]) interpret Eq. (55) as a Doppler shift factor.

Now, suppose we multiply each term in Eq. (46) by $2p_1q_0/\rho_0 q a_0^2$ and collect terms. Since q and the ambient quantities have no time dependence, the result is easily reduced to

$$\frac{\partial E}{\partial t} + \nabla \cdot (E\mathbf{v}_s) = 0 \ . \tag{56}$$

This is the energy conservation equation of linear geometric acoustics. To the extent that E does indeed represent the energy content of the disturbance, $E\mathbf{v}_s$ is the acoustic energy flux, *and is directed along* $\mathbf{v}_s$. This says that the energy of the disturbance is propagated in the direction $\mathbf{v}_s = \mathbf{v}_0 + a_0\hat{\mathbf{n}}$.

These are also the results of the work of Ryshov and Shefter,[21] who obtain equations formally identical to those of Blokhintsev in terms of the acoustic variables, rather than their spectral amplitudes. Ryshov and Shefter apply the fluid equations for instantaneous conservation of energy directly to a wavefront carrying a narrow pulse, but for which the small amplitude acoustic relations and correct equations of the acoustic ray paths are known *a priori*. Landau and Lifshitz[22] also indicate this equivalence for the case of sound energy flux in otherwise motionless ambient media.

Hayes[23] shows that the results of Guiraud,[8] Blokhintsev,[11,12] and Ryshov and Shefter[21] are special cases of a more general formulation of linear acoustic energy propagation in unsteady media, in terms of conservation of adiabatic invariants. It is interesting to note that Hayes also implicitly assumes that these conservation results are applicable to the details of the pulse shape.

We conclude this section with a remark on acoustic perturbation shapes. The intrinsic acoustic perturbation velocity $\mathbf{v}_1$ is along the wavefront normal $\hat{\mathbf{n}}$, but is *not* necessarily along the direction of energy flow, $\mathbf{v}_s$. Thus, the phase surfaces will have a tangential component of motion. What this means for the appearance of the acoustic perturbation is that in the direction of actual propagation, i.e., along $\mathbf{v}_s$, the spatial profile of the perturbation will appear "stretched" relative to that seen "normally," i.e., along $\hat{\mathbf{n}}$. This is because $\mathbf{v}_s$ crosses the surfaces of constant phase at an inclination that is other than perpendicular.

ACOUSTIC RAYS AND RAY TUBE PROPAGATION

In this section we shall determine the differential equations which characterize the paths taken by different points on the wavefront, i.e., the acoustic rays, and then apply these and Blokhintsev's results to obtain the equations of propagation along ray tubes.

Acoustic ray tracing has been a perennially popular topic since Rayleigh[24] determined the correct equation for the alignment of an acoustic wavefront in a vertically stratified, temperature- and wind-loaded atmosphere. Milne[25] was apparently the first to obtain the ray equations for the propagation of an acoustic wavefront in arbitrary, steadily moving atmospheres. His approach was based on applying Huygens' principle to determine successive positions of the front; locally, this front moved with a velocity composed of the local sound speed perpendicular to itself, and the ambient wind velocity. He also derived the differential equation for this surface; his resultant velocity and the differential equation are identical to $\mathbf{v}_s$ and the eikonal equation for $\nabla\theta$ derived in the previous section. The subsequent literature on acoustic ray tracing has since provided variants on Milne's equations and the way they are derived. Engelke[26] reviews some of this work, as well as those papers which extend ray tracing to motion in unsteady ambient fluids. Thompson,[27] who obtains the eikonal and ray equations from the theory of characteristics, gives useful references.

To obtain the equations for the acoustic rays from the eikonal Eq. (32), we could take them directly from Milne,[24] since his equations for the phase surface are equivalent to Eq. (32). Whitham[2] has indicated a more direct procedure from the theory of partial differential equations, which we will employ instead. From a first-order partial differential equation satisfied by a function f, and which takes the form

$$H\left(f, x_i, \frac{\partial f}{\partial x_j}\right) = 0 \ , \tag{57}$$

the following set of 2n+1 ordinary differential equations is obtained:

$$\frac{dp_i}{d\lambda} = -p_i \frac{\partial H}{\partial f} - \frac{\partial H}{\partial x_i} \ , \tag{58}$$

$$\frac{dx_i}{d\lambda} = \frac{\partial H}{\partial p_i} , \tag{59}$$

and

$$\frac{df}{d\lambda} = p_i \frac{\partial H}{\partial p_i} \ , \tag{60}$$

where the x_i are the independent variables, $p_i = \partial f/\partial x_i$, λ is an arbitrary parameter, and the index i runs from 1 to n. Equations (58) and (59) determine a parametric curve $x_i(\lambda)$, along which Eq. (60) can be integrated to obtain f.

We express the eikonal Eq. (32) in the form given by Eq. (57), by letting x_i be the Cartesian coordinate and f be the phase surface function θ/q_0, and defining $p_i = (1/q_0)(\partial\theta/\partial x_i) = \partial f/\partial x_i$. If we represent the components of $\mathbf{v}_0$ by v_i, we obtain, from Eq. (32), the equivalent form (summation convention understood)

$$H = a_0^2 p_i p_i - (1 - v_j p_j)^2 = 0 \ , \tag{61}$$

from which we obtain

$$\frac{dp_i}{d\lambda} = -2\left[a_0 p_j p_j \frac{\partial a_0}{\partial x_i} + (1 - v_k p_k)\, p_m \frac{\partial v_m}{\partial x_i}\right] \ , \tag{62}$$

$$\frac{dx_i}{d\lambda} = 2\left[a_0^2 p_i + (1 - v_k p_k) v_i\right] . \tag{63}$$

The components of the unit vector $\hat{\mathbf{n}}$ normal to the wavefront are given by $n_i = p_i/p$, where $p^2 = p_i p_i$. On making use of this and Eq. (61), we obtain from Eqs. (62) and (63)

$$\frac{dp_i}{d\lambda} = -2a_0 p^2 \left(\frac{\partial a_0}{\partial x_i} + n_m \frac{\partial v_m}{\partial x_i}\right) \tag{64}$$

$$\frac{dx_i}{d\lambda} = 2a_0 p\,(a_0 n_i + v_i) \ . \tag{65}$$

Since we are ultimately interested in the variation of the direction of $\hat{\mathbf{n}}$ as the wavefront progresses, we replace p_i by $n_i p$ in Eq. (64) to get the result

$$\frac{dn_i}{d\lambda} = 2a_0 p\,(n_i n_j - \delta_{ij}) \left(\frac{\partial a_0}{\partial x_j} + n_m \frac{\partial v_m}{\partial x_j}\right) \ . \tag{66}$$

As we have previously shown, $\mathbf{v}_s$ is the velocity with which a point on the wavefront travels; we then necessarily have

$$\frac{dx_i}{dt} = v_i + a_0 n_i \ . \tag{67}$$

On comparing Eqs. (65) and (67), we choose λ to be such that $2a_0 p \, d\lambda = dt$. Then Eqs. (65) and (67) are the same, and Eq. (66) becomes

$$\frac{dn_i}{dt} = (n_i n_j - \delta_{ij}) \left(\frac{\partial a_0}{\partial x_j} + n_m \frac{\partial v_m}{\partial x_j}\right) \ . \tag{68}$$

Equations (67) and (68) are the desired differential equations of the acoustic rays: the x_i locate a point on the ray path, and the n_i determine the orientation of the wavefront surface; each of these quantities is obtained by integration. From Eq. (68) we see that the rate at which the rays bend depends directly on the presence of gradients in the ambient sound speed a_0 and (in the case of the atmosphere) the ambient winds $\mathbf{v}_0$. These equations are in the form given by Ryshov and Shefter,[21] and are equivalent to those derived by Milne.[25] Gubkin[28] derives these equations directly from the basic hydrodynamic equations without use of the eikonal.

A ray tube is simply a cylindrical surface generated by the acoustic rays, in which the axis and walls are locally parallel to $\mathbf{v}_s$. If Eq. (56) is applied by means of the divergence theorem to a portion of such a tube, we see that the energy flux through any cross section of the tube is a constant, that is, $\int E\mathbf{v}_s \cdot d\mathbf{A}$ is invariant. For tubes of small diameter, the quantity Ev_sA is constant; A is the *normal* cross-section area of the tube, which does not necessarily coincide with the wavefront surface. Thus, knowledge of how this area varies is sufficient to determine how the amplitude of a pulse changes as the pulse propagates, once this amplitude is known at any given place on the tube. This is the basis of the method of obtaining acoustic intensities in a moving medium, as given by Ryshov and Shefter,[21] and as exemplified by the work of Thompson[29] and Ugincius[30]; a practical extension of this method was employed by Candel[31] in computer calculations.

This conservation approach does not lend itself easily to calculations that depend on the detailed shape of the acoustic disturbance, as we will need for our nonlinear extensions. Instead, we will use Eq. (46), but in terms of the scalar magnitude of the perturbation velocity $\mathbf{v}_1$. Accordingly, from Eqs. (47) and (55), Eq. (54) becomes

$$E = \left(1 + \frac{v_n}{a_0}\right) \rho_0 v_1^2 \tag{69}$$

and the propagation Eq. (46) becomes

$$\frac{\partial v_1}{\partial t} + \mathbf{v}_s \cdot \nabla v_1 + \frac{1}{2} \left\{ \nabla \cdot \mathbf{v}_s + \mathbf{v}_s \cdot \nabla \ln \left[\rho_0 \left(1 + \frac{v_n}{a_0}\right)\right]\right\} v_1 = 0 \ . \tag{70}$$

We now put the spatial derivatives in terms of distance along the ray tube, which we shall denote by s. Since $\mathbf{v}_s$ lies along the tube axis, the operator $\mathbf{v}_s \cdot \nabla$ becomes $v_s \, \partial/\partial s$; by applying the divergence theorem to a small portion of the ray tube, we obtain, if the tube portion is sufficiently narrow,

$$\nabla \cdot \mathbf{v}_s = \frac{\partial v_s}{\partial s} + \frac{v_s}{A} \frac{\partial A}{\partial s} = v_s \frac{\partial}{\partial s} \ln(v_s A) \ . \tag{71}$$

In terms of the ray tube axial coordinate, then, Eq. (70) becomes

$$\frac{\partial v_1}{\partial t} + v_s \frac{\partial v_1}{\partial s} + \frac{1}{2}\left\{ v_s \frac{\partial}{\partial s} \ln \left[\rho_0 v_s A \left(1 + \frac{v_n}{a_0}\right)\right]\right\} v_1 = 0 \ . \tag{72}$$

This is the linear equation of propagation in the desired one-dimensional form. Here, v_s is the magnitude $|\mathbf{v}_0 + a_0 \hat{\mathbf{n}}|$ and v_1 is the magnitude of $\mathbf{v}_1'$; thus we are calculating $\mathbf{v}_1$ even though its direction is along $\hat{\mathbf{n}}$.

Equations (67), (68), and (72) are in a form that enables them to be solved quite readily by digital computer methods: Eqs. (67) and (68) are ordinary differential equations, while Eq. (72) can be put in the form

$$\frac{dv_1}{dt} = -\frac{1}{2}\left\{ v_s \frac{\partial}{\partial s} \ln \left[\rho_0 v_s A \left(1 + \frac{v_n}{a_0}\right)\right]\right\} v_1 \tag{73}$$

in the rest frame of the disturbance. Once the ray paths are calculated, not only can the area A be found from at least three closely spaced rays, but the magnitude v_s can be determined as well.

NONLINEAR EXTENSIONS

The linear theory we have established so far is applicable where convective, dissipative, and otherwise dispersive effects are negligible. In practice, this would correspond to acoustic propagation over short distances in a neutral atmosphere. As we have shown, the ambient fluid has two effects on the propagating waveform: it amplifies or diminishes its magnitude, and it determines where it goes by means of the ray paths. Note that if the disturbance grows as it propagates, it can still remain acoustic, i.e., small relative to the ambient fluid. For example, a wave travelling in the real atmosphere is known to increase in magnitude as it encounters rarefied air; on the other hand the speed of sound is also increasing, so that the relative strength of the wave, denoted by v_1/a_0, does not increase as markedly.

For the situation where the acoustic propagation extends over long distances, the dissipative and second-order convective terms that were discarded will produce a substantial cumulative effect, *even in the case where the disturbance remains acoustic in magnitude*. This is manifested as a change in the profile of the disturbance. Thus, it is not necessary that the magnitude of the disturbance be large for nonlinear behavior to be present. In the remainder of this paper, we will be concerned specifically with those nonlinear changes of shape for disturbances that stay acoustically small. By making this important distinction, we are able to retain all the results that have been developed so far for the linear case. In particular, the ambient fluid will remain essentially unaffected by the passage of the disturbance, and the ray tube geometry will still apply. This last assumption has already been justified by Heller,[17] for the case where nonlinear convection is present, and shown by Gubkin[28] to hold for weak shock waves. Dissipation for the most part reduces the amplitude of the disturbance, and this does not conflict with the assumption of acoustic smallness.

The consequences of these combined nonlinear effects have been treated in detail by Lighthill[32] for planar disturbances, and by Naugol'nykh[33] for spherical disturbances, where the propagation is into a uniform, viscous, heat-conducting, and motionless medium. A more recent review of these matters, including extensions to sound waves of large amplitude, is contained in the treatise of Rudenko and Soluyan.[34] In this section we shall draw primarily on the last two references for estimating the form of the dissipative terms, after which we shall treat the second-order convective terms in a manner similar to that of Lighthill.[32]

To keep these matters in a manageable form, we shall initially assume that the ambient fluid is motionless and uniform. Our plan is to elucidate the nature of the nonlinear terms, and then carry them over to the equation for propagation in the inhomogeneous, moving fluid, by the simple expedient of employing a local, almost Galilean transformation. This should not affect the essential validity of the overall results, since Blokhintsev's treatment has already accounted for the effects of the ambient gradients and flow fields. We are thus assuming that the nonlinear actions are basically unaffected by the ambient inhomogeneity and flow. Such a procedure is clearly inappropriate for finite amplitude disturbances, i.e., those that are much larger than acoustic.

The dissipative effects are treated as corrections to what is regarded as an approximately isentropic situation, in agreement with the view that they become evident only after the propagation has lasted for a long time. This is done by analyzing the entropy and momentum losses sustained by the disturbance. Blokhintsev's Eqs. (9)–(11) for a motionless, uniform medium, with just the dissipative terms restored, are

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \ , \tag{74}$$

$$\frac{\partial \mathbf{v}_1}{\partial t} + \frac{1}{\rho_0} \nabla p_1 = \frac{1}{\rho_0} \nabla \vec{\vec{\sigma}}_1 \ , \tag{75}$$

and

$$\frac{\partial s_1}{\partial t} = \frac{\kappa}{\rho_0 T_0} \nabla^2 T_1 \ . \tag{76}$$

These are the usual equations of linear acoustics; the last one is the equation of heat flow. Now, since we are assuming the propagation is approximately isentropic, we have

$$T_1 = \left(\frac{\partial T}{\partial p}\right)_s p_1 \ , \tag{77}$$

and also, from Eq. (75),

$$\frac{\partial \mathbf{v}_1}{\partial t} \cong -\frac{1}{\rho_0} \nabla p_1 \ . \tag{78}$$

We insert Eq. (77) into Eq. (76), take the divergence of Eq. (78), and eliminate $\nabla^2 p_1$ from the result. Then we obtain

$$\frac{\partial s_1}{\partial t} = -\frac{\kappa}{T_0}\left(\frac{\partial T}{\partial p}\right)_s \frac{\partial}{\partial t} (\nabla \cdot \mathbf{v}_1) \ , \tag{79}$$

from which we infer that, for linear acoustic disturbances,

$$s_1 = -\frac{\kappa}{T_0}\left(\frac{\partial T}{\partial p}\right)_s \nabla \cdot \mathbf{v}_1 \ . \tag{80}$$

This is easily seen by considering sinusoidal disturbances. As a result, the entropy contribution to the "perturbative equation of state" Eq. (16) becomes

$$h_0 s_1 = -\frac{\kappa}{T_0}\left(\frac{\partial p}{\partial s}\right)_\rho \left(\frac{\partial T}{\partial p}\right)_s \nabla \cdot \mathbf{v}_1 \ , \tag{81}$$

or, by use of standard thermodynamic identities,[35]

$$h_0 s_1 = -\left(\frac{1}{c_v} - \frac{1}{c_p}\right) \kappa \nabla \cdot \mathbf{v}_1 \ . \tag{82}$$

Next, we consider the equation of state. For reasons that will be apparent shortly, we carry the

Taylor expansion represented by Eq. (16) to second order in ρ_1, and obtain

$$p_1 = h_0 s_1 + a_0^2 \rho_1 + \frac{1}{2}\left(\frac{\partial a_0^2}{\partial \rho}\right)\rho_1^2 . \tag{83}$$

Taking the gradient, we obtain

$$\nabla p_1 = h_0 \nabla s_1 + \left[a_0^2 + \left(\frac{\partial a_0^2}{\partial \rho}\right)\rho_1\right]\nabla \rho_1 . \tag{84}$$

The quantity in brackets can conveniently be regarded as an expansion of the square of the local adiabatic speed of sound in the fluid when the perturbation is present, i.e., when $\rho = \rho_0 + \rho_1$. This is because for an adiabatic process, a^2 is essentially a function only of ρ, and we may identify the acoustic increment on a_0^2 with $(\partial a_0^2/\partial\rho)\rho_1$. On making this substitution and using Eq. (82), we obtain the approximate relation

$$\nabla p_1 \cong a^2 \nabla \rho_1 - \left(\frac{1}{c_v} - \frac{1}{c_p}\right)\kappa \nabla(\nabla \cdot \mathbf{v}_1) , \tag{85}$$

which we will need further on.

Finally, we put the viscosity term $\nabla\overleftrightarrow{\sigma}_1$ in standard vector form, and obtain

$$\nabla\overleftrightarrow{\sigma}_1 = \left(\zeta + \frac{1}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v}_1) + \eta \nabla^2 \mathbf{v}_1 . \tag{86}$$

To the extent that $\nabla \times (\nabla \times \mathbf{v}_1)$ is vanishingly small, which would be true for locally planar disturbances (particularly the pulsed wavefront of the previous section), the grad div and Laplacian operators can be used interchangeably. Thus, e.g.,

$$\nabla\overleftrightarrow{\sigma}_1 = \left(\zeta + \frac{4}{3}\eta\right)\nabla(\nabla \cdot \mathbf{v}_1) . \tag{87}$$

We now deal with the nonlinear extensions and dissipation corrections for the acoustic propagation. To see what these terms should be, we consider the full equations for an almost isentropic disturbance propagating into a motionless, uniform fluid in which viscosity and heat conduction are present. There are no body forces or ambient gradients, and the momentum Eq. (2) becomes

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\nabla\overleftrightarrow{\sigma} \tag{88}$$

where $\mathbf{v}$ and the gradients are now of the perturbation, i.e., $\nabla p = \nabla p_1$, $\nabla\rho = \nabla\rho_1$, etc. Then Eqs. (85) and (87) can be used in Eq. (88) to obtain

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{a^2}{\rho}\nabla\rho = \delta \nabla(\nabla \cdot \mathbf{v}) , \tag{89}$$

where

$$\delta = \frac{1}{\rho_0}\left[\zeta + \frac{4}{3}\eta + \kappa\left(\frac{1}{c_v} - \frac{1}{c_p}\right)\right] . \tag{90}$$

All dissipative effects of interest have thus been assembled into the momentum equation; this can be regarded as the three-dimensional generalization of Lighthill's result.[32] Equation (90) describes the acoustic coefficient of absorption in one of its more familiar forms.[22,32]

Let the fluid now be constrained to a long, narrow tube of variable cross section, and let the disturbance propagate along the axis of this tube. The equations of motion for the disturbance are given by Eqs. (1) and (89) applied to the tube. Since $\mathbf{v}$ is along the tube axis, the divergence terms may be expanded as in Eq. (71); this results in

$$\frac{\partial\rho}{\partial t} + v\frac{\partial\rho}{\partial s} + \rho\frac{\partial v}{\partial s} = -\frac{\rho v}{A}\frac{\partial A}{\partial s} , \tag{91}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial s} + \frac{a^2}{\rho}\frac{\partial\rho}{\partial s} = \delta\frac{\partial}{\partial s}\left[v\frac{\partial}{\partial s}\ln(vA)\right] , \tag{92}$$

where s is the coordinate along the tube, A is its cross section, and v is the magnitude of the disturbance. Unlike the ray tube case, the surfaces of constant phase (or constant disturbance) are, here, normal to the tube axis. Because we have a uniform ambient fluid, Riemann's method of solution can easily be applied to these equations, even though they are inhomogeneous. (Cf. Whitham[36] and Lighthill[32]). We multiply Eq. (91) by a/ρ, add and subtract the result from Eq. (92), to obtain

$$\left[\frac{\partial}{\partial t} + (v \pm a)\frac{\partial}{\partial s}\right]J_\pm = \mp\frac{va}{A}\frac{\partial A}{\partial s} + D , \tag{93}$$

where D symbolizes the right-hand side of Eq. (92). The J's are the Riemann variables

$$J_{\pm} = v \pm \int \frac{a}{\rho} d\rho \ , \tag{94}$$

and the ambiguous sign symbol indicates how these equations are to be set up for propagation in the opposite directions along the tube.

Now, if, as we stipulated, the dissipation is small, and if the tube area variation is also small, then the $J_{\pm}$ in Eq. (93) may be regarded as approximately invariant as they propagate. To have a disturbance travelling only in the +s direction, we demand that $J_{-} = 0$, namely

$$v = \int_{\rho_0}^{\rho} \frac{a}{\rho} d\rho \ . \tag{95}$$

Thus $J_{+} = 2v$, and our desired equation with the nonlinear terms is now

$$\frac{\partial v}{\partial t} + (a + v)\frac{\partial v}{\partial s} = -\frac{av}{2}\frac{\partial}{\partial s}\ln A + \frac{1}{2}D \ . \tag{96}$$

For straight tubes of constant cross section we have planar propagation, and Eq. (96) reduces to the form obtained by Lighthill.[32] In the acoustic limit, Eq. (95) gives $v = a_0\rho_1/\rho_0$, the familiar small amplitude relation.

As is well known, the nonlinearity of Eq. (96) is evident. The total coefficient of $\partial v/\partial s$ gives the speed at which each part of the disturbance, of magnitude v, moves; parts with larger values will move faster because of the (v + a) contribution, and this speed will be cut back because of the dissipative contribution. Because these actions do not evenly balance, the profile of the disturbance will be progressively distorted as it moves. As Lighthill has shown,[32] the dissipative effects can prevent a shock from ever forming. For infinitesimally small disturbances, Eq. (96) reduces to the small-amplitude acoustic result,

$$\frac{\partial v}{\partial t} + a_0\frac{\partial v}{\partial s} = -\frac{a_0 v}{2}\frac{\partial}{\partial s}\ln A + \frac{1}{2}D \ . \tag{97}$$

At this point we are ready to consider the main question of how to incorporate these nonlinear effects into the ray-tube propagation process. We do this by determining what the nonlinear flow (Eqs. (96) and (97)) looks like when the ambient fluid is moving. For this purpose, Eq. (72), rewritten for uniform ambient fluids, may be regarded as the result of an almost Galilean transformation to a frame of reference in which the ambient fluid is moving. Equation (72) becomes

$$\frac{\partial v_1}{\partial t} + v_s\frac{\partial v_1}{\partial s} = -\frac{v_s v_1}{2}\frac{\partial}{\partial s}\ln\left[v_s A\left(1 + \frac{v_n}{a_0}\right)\right] \ , \tag{98}$$

where we recall that $\mathbf{v}_s = \mathbf{v}_0 + a_0\hat{\mathbf{n}}$ and a_0 is now a constant. If $\mathbf{v}_0 = 0$, Eq. (98) reduces to Eq. (97), apart from the dissipative term. We would therefore expect Eq. (98) to be the result of applying underlying ambient fluid flow to Eq. (97), provided we identify the tubes associated with the nonlinear scenario as acoustic ray tubes in a motionless ambient medium.

Now, during the incorporation of nonlinear effects, the geometry of the situation must be kept in mind; recall that the ambient flow field tilts the wavefront normal with respect to the direction of actual propagation. This is important, because in the local frame of reference moving with the ambient flow, the nonlinear processes will be assumed to still proceed in the direction perpendicular to the wavefront. That is, in the moving fluid, *the nonlinear extensions will be associated, to a first approximation, with the vector normal* $\hat{\mathbf{n}}$, in accord with the approximately Galilean aspect of the shift in the local frame of reference.

We shall therefore "nonlinearize" Eq. (72) as follows: the v_s on the left-hand side is replaced by

$$v_s' = |\mathbf{v}_0 + (a + v_1)\hat{\mathbf{n}}| \tag{99}$$

to account for the nonlinear convection; this is reminiscent of Heller's result.[17] The first occurrence of v_s on the right-hand side is replaced by

$$v_s'' = |\mathbf{v}_0 + a\hat{\mathbf{n}}| \ . \tag{100}$$

To see this, compare the usage of a, a_0 and (a + v) in Eqs. (98), (97), and (96). Insofar as the dissipative term in Eq. (92) is concerned, we shall ignore the refinements associated with the fact that $\partial \ln A/\partial s$ undergoes subtle changes when carried over to the moving fluid. Thus, we directly add on the dissipative term. We then obtain our main result, which is the nonlinearized ray tube propagation equation

$$\frac{\partial v_1}{\partial t} + v_s' \frac{\partial v_1}{\partial s}$$

$$= -\frac{1}{2}\left\{v_s'' \frac{\partial}{\partial s} \ln\left[\rho_0 v_s A \left(1 + \frac{v_n}{a_0}\right)\right]\right\} v_1$$

$$+ \frac{1}{2}\delta \frac{\partial}{\partial s}\left[v_1 \frac{\partial}{\partial s} \ln(v_1 A)\right] , \qquad (101)$$

where v_s', v_s'', and δ are as given above, and where ρ_0 carries additional information regarding the nonuniformity of the medium.

For practical acoustic purposes, we could take $v_s'' = v_s$ and $a = a_0$ in the terms on the right-hand side of Eq. (101); however, v_s' must be retained almost exactly as it is, to account for the nonlinear convective effects. If we write $a = (a - a_0) + a_0$ in Eqs. (99) and (100), and expand the right-hand side binomially, we get the acoustically acceptable approximations

$$v_s' = v_s + \left(\frac{v_n + a_0}{v_s}\right)(a - a_0 + v_1) , \qquad (102)$$

$$v_s'' = v_s + \left(\frac{v_n + a_0}{v_s}\right)(a - a_0) . \qquad (103)$$

For small ambient flow the first term in parentheses is practically unity. It is interesting to notice that the effect of the ambient flow is to slightly reduce or enhance the nonlinear terms, depending on whether the flow is with or against the direction of the wavefront normal.

APPLICATION TO A GRAVITATIONALLY STRATIFIED MODEL ATMOSPHERE

The real terrestrial atmosphere is gravitationally stratified, and admits of extremely complex motions.[3] Its composition and properties change dramatically with geographical location and altitude, and these, in turn, have periodic variations that range from the diurnal to the annual.[37] From the point of view of a propagating acoustic pulse, these time variations are extremely slow. The speed of sound has a range from about 330 m/s at sea level to about 700 m/s at F-region ionospheric heights, and an acoustic pulse will take about 10 to 12 min to cover this vertical distance. The same order of time is also required for a ground-launched pulse traveling at a low elevation angle to reach apogee before returning to the ground some 200 to 300 km distant.[38] Thus, apart from the phenomenon of ducting,[2,3] the free propagation of an acoustic pulse has a fairly limited range, as distinguished from that of acoustic-gravity waves and other larger scale wave motions that characterize the atmosphere as a whole.

Over these durations and distances, the atmosphere may be regarded as approximately uniform in the horizontal direction and in a steady state, and the earth as flat. For purposes of illustrating the theoretical material developed in the previous section, we shall represent the atmosphere with the following idealized model: all fluid quantities, physical properties, and horizontal wind magnitudes and directions are vertically stratified, i.e., are functions only of the altitude z above the surface; there are no vertical wind components. The equation of state is given by the polytropic ideal fluid, i.e., with constant specific heats. Near the ionospheric regions and above, this model is not very good, as the implied assumption of an isotropic, neutral and continuous fluid begins to break down. As long as the spatial extent of the disturbance continues to be larger than the molecular mean free path, and as long as the equilibrium molecular-interaction processes occur faster than the acoustic fluid dynamic processes, this model should be reasonably adequate.

In terms of this model, the ambient fluid properties are related, through Eqs. (6) and (7), by the familiar hydrostatic condition

$$\frac{1}{\rho_0}\nabla p_0 = g(z) \tag{104}$$

since all terms involving $\nabla \cdot \mathbf{v}_0$ and $\mathbf{v}_0 \cdot \nabla$ have vanished. The polytropic equation of state is

$$\frac{p}{\rho} = (\gamma - 1)\epsilon = RT \ , \tag{105}$$

where $\gamma = c_p/c_v$ and $R = c_p - c_v$. As a consequence, the entropy function at each point in the atmosphere is given by

$$s(z) = c_v \ln p(z) - c_p \ln \rho(z) \ . \tag{106}$$

In a sense this is an assumption in its own right, since the real atmosphere has distributed thermal and chemical sources and sinks.

We remark in passing that this model allows the ambient fluid to have vortex components, i.e., $\nabla \times \mathbf{v}_0$ is not necessarily zero. It is not so obvious that the acoustic disturbance field can also be rotational; this is really a consequence of Blokhintsev's equations, rather than the choice of the ambient conditions. From Eqs. (47) and (48), $\nabla \times \mathbf{v}_1 = \nabla \ln(p_1/\rho_0 q) \times \mathbf{v}_1$, and unless the disturbance propagates specifically along the atmospheric gradients, it is necessarily rotational, even when $\mathbf{v}_0 = 0$.

The vertical stratification simplifies the acoustic ray equations enormously. We choose a Cartesian coordinate system with the z-axis vertical, and let the indices 1, 2, 3 represent the x, y, and z directions respectively. The components of the wavefront normal n_1, n_2, and n_3 are given by $\sin\theta\cos\phi$, $\sin\theta\sin\phi$, and $\cos\theta$, respectively, where θ and ϕ are the usual spherical polar and azimuthal angles measured correspondingly from the z- and x-axes, and the components of the wind velocity $\mathbf{v}_0$ are represented as v_x, v_y and v_z. In these terms, Eqs. (67) and (68) for the model atmosphere reduce to

$$\frac{dx}{dt} = v_x + a_0 \sin\theta\cos\phi \ , \tag{107}$$

$$\frac{dy}{dt} = v_y + a_0 \sin\theta\sin\phi \ , \tag{108}$$

$$\frac{dz}{dt} = a_0 \cos\theta \ , \tag{109}$$

and

$$\frac{d\theta}{dt} = \sin\theta\left[\frac{\partial a_0}{\partial z} + \sin\theta\left(\cos\phi\frac{\partial v_x}{\partial z} + \sin\phi\frac{\partial v_y}{\partial z}\right)\right] \ , \tag{110}$$

where a_0 is now the ambient polytropic speed of sound, given from Eq. (106) by

$$a_0^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = \frac{\gamma p_0}{\rho_0} \ . \tag{111}$$

These are also Milne's[25] results; Milne noted the remarkable fact that these equations predict that the azimuthal orientation of the wavefront normal along any given ray path does not change during the propagation. After all, there are no time derivatives of ϕ involved.

It is amusing to take note of a mild controversy that has historically pervaded the literature[39] with respect to Rayleigh's original investigation of the refraction of sound by horizontal winds,[24] in which he correctly derives an expression for the wavefront normal angle, θ, Thompson's remarks to the contrary[27] notwithstanding. Indeed, Rayleigh's result is consistent with Eq. (110). Thompson and others[39] are, however, correct in pointing out that Rayleigh did not distinguish between the direction of the wave normal and the direction of actual propagation, in his subsequent work on the matter.

The propagation equation for this model is likewise simply obtained, by eliminating the explicit dependence on ρ_0 from Eq. (101), and determining the functional form of a(v) from Eq. (95). Taking the gradient of a_0 in Eq. (111), and substituting for ∇p_0 from Eq. (104), we obtain

$$\nabla \ln \rho_0 = \frac{\gamma \mathbf{g}}{a_0^2} - \nabla \ln a_0^2 \ . \tag{112}$$

On taking the scalar product with $\mathbf{v}_s$ we get

$$\frac{\partial}{\partial s}\ln\rho_0 = -\frac{\gamma g\cos\theta_s}{a_0^2} - \frac{\partial}{\partial s}\ln a_0^2 \tag{113}$$

where $-g \cos\theta_s$ is the component of the downward gravitational acceleration along the direction of propagation, and θ_s is the polar angle for this direction.

Since the perturbation in the nonlinear treatment is assumed to propagate isentropically, it follows from Eq. (106) that

$$a^2 = \left(\frac{\partial p}{\partial \rho}\right)_s = a_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1} , \tag{114}$$

and this is the functional form of $a(\rho)$ required by the integral in Eq. (95). On carrying out the integration, we obtain the familiar result

$$a = a_0 + \frac{\gamma - 1}{2} v_1 , \tag{115}$$

which is exact. After inserting this into Eqs. (102) and (103), we have

$$v_s' = v_s + \left(\frac{v_n + a_0}{v_s}\right) \frac{\gamma + 1}{2} v_1 , \tag{116}$$

$$v_s'' = v_s + \left(\frac{v_n + a_0}{v_s}\right) \frac{\gamma - 1}{2} v_1 . \tag{117}$$

With these substitutions and Eq. (113), Eq. (101) becomes

$$\frac{\partial v_1}{\partial t} + v_s' \frac{\partial v_1}{\partial s} = \frac{1}{2} v_s'' v_1 \left\{\frac{\gamma g \cos\theta_s}{a_0^2} - \frac{\partial}{\partial s} \ln\left[\frac{v_s A}{a_0^2}\left(1 + \frac{v_n}{a_0}\right)\right]\right\} + \frac{1}{2}\delta \frac{\partial}{\partial s}\left[v_1 \frac{\partial}{\partial s} \ln(v_1 A)\right] , \tag{118}$$

and this is our model result. Note that the gradients of the stratified ambient quantities are given by $\partial/\partial s = \cos\theta_s \, \partial/\partial z$. The ray path and ray tube equations now depend on just four variable properties of the model atmosphere: the local sound speed $a_0(z)$, the horizontal wind velocity $v_0(z)$, the acceleration due to gravity, $g(z)$, and the kinematic viscosity imbedded in $\delta(z)$.

A variety of different algebraic forms can be taken by Eq. (118); its compact simplicity is highly deceptive, because the notation $\nabla \ln f$ is a convenient shorthand for either $\nabla f/f$ or $\nabla \ln f/f_0$, where f_0 is a reference constant value. As a useful illustrative example, we consider the form Eq. (118) would take if there were no ambient winds. Then $v_s = a_0$, $v_n = 0$, $\theta_s = \theta$, and we obtain

$$\frac{\partial v_1}{\partial t} + \left(a_0 + \frac{\gamma + 1}{2} v_1\right)\frac{\partial v_1}{\partial s} = \frac{1}{2} v_1 \left[1 + \frac{\gamma - 1}{2}\left(\frac{v_1}{a_0}\right)\right]\left[\left(\frac{\gamma g}{a_0} + \frac{\partial a_0}{\partial z}\right)\cos\theta - \frac{a_0}{A}\frac{\partial A}{\partial s}\right] + \frac{1}{2}\delta \frac{\partial}{\partial s}\left[v_1 \frac{\partial}{\partial s} \ln(v_1 A)\right] . \tag{119}$$

The first term in the brackets is essentially a nonlinear correction to the exponential growth factor,[36] and the left-hand side is in the familiar nonlinear wave propagation form.

CONCLUDING REMARKS

The main results of this paper are the ray path Eqs. (67) and (68), and the nonlinear propagation equation collectively represented in the general case by Eqs. (90), (101), (102), and (103). For application to the model terrestrial atmosphere, these last would be Eqs. (90), (116), (117), and (118). In this form, the equations are quite amenable to solution as initial value problems by using contemporary numerical methods; Dubois[40] has in fact already set up such a scheme for the model atmosphere, and we will report on this in a future publication.

These results, although quite general in nature, are nevertheless an asymptotic "high frequency" approximation, and, moreover, are applicable primarily to disturbances that are acoustically small in magnitude. A more proper theory must necessarily couple the ray path and propagation equations. This has been done for weak shock waves (see, e.g., Whitham[2]), and future work leading to such a theory could conceivably take guidance from this area. The work of Friedman, Kane, and Sigalla[7] is a potential example in point.

It is possible to improve the one-dimensional nonlinear treatment by carrying out Riemann's procedure for the case in which the motionless ambient fluid is not uniform, and then transforming to the moving fluid. The difficulty in this process lies in getting the fluid equations into a Riemannian form similar to Eqs. (93) and (94), because the integral in Eq. (94) is not as easily extracted for this case as it was for the case of a uniform ambient fluid, and the entropy state function must now include a dependence on v. We have carried out such a calculation for the special case of the motionless model atmosphere of the previous section, and the results are as given by Eq. (119), except for minor variations in the second-order coefficients. That is, the coefficients of the (v_1/a_0) terms associated with $\gamma g/a_0$ and $\partial a_0/\partial z$ turned out respectively to be $(\gamma - 1)(\gamma + 3)/4\gamma$ and $(\gamma + 3)/2$ instead of $(\gamma - 1)/2$ for both.

We note at this point that to obtain the other fluid variables from the solution v_1 of Eqs. (101) or (118), the small-amplitude acoustic relations $\rho_1 = \rho_0 v_1/a_0$ and $p_1 = \rho_0 a_0 v_1$ are adequate. If, however, the disturbance becomes large, then the full fluid variables p and ρ should be expanded in powers of v_1/a_0 with the help of Eq. (115) and the adiabatic relationship between p and ρ. These expansions are easy to do, since, e.g., $\rho/\rho_0 = 1 + \rho_1/\rho_0$.

A quick survey of recent literature reveals other potential areas to which the nonlinear methods of this paper could profitably be extended. For example, ray equations have already been used to determine the paths taken by sound waves in well-defined vortices, as given by the work of Georges[41] and Broadbent[42]; Blythe[43] has investigated the use of Riemann invariants in gases with vibrational relaxation rates comparable to those of the hydrodynamic processes; Broer[44] has developed the method of characteristics for a chemically reacting gas; and Panchev and Pancheva[45] have equations corresponding to the propagation of sound pulses in electrically conducting terrestrial atmospheres, i.e., the ionosphere. Whitham[46] gives a comprehensive treatment of such extensions in the case of magnetohydrodynamic wave motions. While shock wave ray and propagation methods are well established for some of these areas,[2] the "shockless" techniques appear not to have been as well developed.

In conclusion, we note that the methods of this paper could conceivably be extended to time-varying media. The discussions by Hayes[23] and Engelke[26] suggest that Blokhintsev's work and Milne's ray path development are also suitable for fluids having unsteady ambient motions.

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